# Reply to Speckman and Rouder:

# A theoretical basis for QML

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Speckman and Rouder (submitted) point out that Heathcote, Brown and Mewhort (2002) did not prove that their quantile-based estimator (QML) approximates likelihood, but found that it outperformed an exact maximum likelihood method based on order statistics for small samples (n=20) from the Ex-Gaussian distribution. When QML uses all order statistics as quantile estimates, it is equivalent to maximum spacings product (MSP) estimation (Cheng & Amin, 1983; Ranneby, 1984), which can be derived from information theory as a measure of model fit (Ekstrom, 2001). MSP has been shown to have asymptotic estimation properties that closely parallel conventional maximum likelihood (CML) for regular problems, and better performance in irregular problems, such as estimation of heavy tailed distributions, mixtures of continuous distributions, and distributions with a minium that must be estimated ("shift" distributions). CML can produce inconsistent estimates under these conditions, in particular for shift distributions commonly used in response time research, such as the Lognormal, Gamma and Weibull, whereas MSP maintains efficiency or is "hyper-efficient"[1].

MSP is apparently obscure; neither we, nor the reviewers of our earlier work, were aware of MSP or the link to QML. We became aware of MSP in research for a paper on estimation software for shift distributions (Heathcote, Brown & Cousineau, submitted). QML generalizes MSP by using linear combinations of order statistics (i.e., empirical quantiles estimates). Titterington (1985) suggested a similar extension of MSP, which uses averages of adjacent order statistics and is equivalent to the QML1 method that Heathcote et al. (2002) found to provide the best performance amongst the special cases of QML they examined.

Cheng and Iles (1987) point out that like MSP, Titterington's (1985) method (and hence QML1) is consistent in irregular problems, and comment that it is a "viable alternative [to MSP], though we have yet to investigate details." (p.99). We performed a preliminary investigation of these details by comparing MSP to QML1 estimates using Speckman and Rouder's (submitted) simulation set-up, but with 20000 replicates. Both MSP and QML1 estimates were obtained by minimising $\sum_{j=2}^{m} \ln(D_j)$, $D_j = \int_{\hat{q}_{j-1}}^{\hat{q}_j} f(t,\mathbf{\Theta})dt$, where $f(t,\mathbf{\Theta})$ is the Ex-Gaussian density at $t$, and Θ=(μ,σ,τ). For MSP, $m$=$n$+2 and $\hat{\mathbf{q}} = (-\infty, x_{(1)}, x_{(2)},..., x_{(n)}, \infty)$. For QML1 $\hat{\mathbf{q}} = (-\infty, (x_{(1)} + x_{(2)})/2, (x_{(2)} + x_{(3)})/2,..., (x_{(n-1)} + x_{(n)})/2, \infty)$ and $m$=$n$+1. In both cases $x_{(i)}$ is the i'th order statistic of the sample. Both methods performed similarly, with MSP slightly less biased (for MSP/QML1, mean bias was μ: 1.8/8.3, σ: 2.8/-1.1, τ: 4.3/-5.9 respectively) but QML1 slightly more efficient (SD($\hat{\mu}$): 36.4/35.4, SD($\hat{\sigma}$): 27.1/25.3, SD($\hat{\tau}$): 44.5/41.2).

We conclude that QML1 is a viable alternative to MSP and that researchers can take advantage of the superior estimation properties of QML1 assured of consistency even in irregular cases. More work is needed to derive results for the efficiency of QML1, and the consistency, efficiency and asymptotic distribution of QML estimates based alternative sets of quantiles[2] (see Brown & Heathcote, in press, for results related to asymptotic distribution). However, we agree with Rouder and Speckman (submitted) that QML does not have an exact theoretical basis in likelihood (although it often approximates likelihood), and might better be named "maximum quantile probability product" (MQP) estimation.

# References

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# Footnotes

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[1]For example, Cheng and Amin (1983) show that for the Gamma and Weibull distributions with shape parameter<2, the minimum parameter estimate has variance smaller than order $n^{-1}$, where $n$ is sample size.

[2] For example, Heathcote et al.'s (2002) QML4 method, with exactly 4 observations falling between each quantile estimate, uses $\hat{\mathbf{q}} = (-\infty, (x_{(4)} + x_{(5)})/2, (x_{(8)} + x_{(9)})/2, \ldots, (x_{(n-1)} + x_{(n)})/2, \infty)$ and $m=(n-1)/4$. In principle, QML may also be used with different quantile estimation algorithms (see Hyndman & Fan, 1996) and uneven quantile intervals.